

A NOTE ON INEXTENSIBLE FLOWS OF PARTIALLY AND PSEUDO NULL CURVES IN E_1^4

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Abstract. In this paper, we study inextensible flows of partially null and pseudo null curves in E_1^4 . We give necessary and sufficient conditions for inextensible flows of partially null and pseudo null curves in E_1^4 .

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1. Introduction

Recently, the study of the motion of inextensible curves has arisen in a number of diverse engineering applications. The flow of a curve is said to be inextensible if the arc length is preserved. Physically, inextensible curve flows give rise to motions in which no strain energy is induced. The swinging motion of a cord of fixed length, for example, or of a piece of paper carried by the wind, can be described by inextensible curve and surface flows. Such motions arise quite naturally in a wide range of physical applications. For example, both Chirikjian and Burdick [6] and Mochiyama et al. [19] study the shape control of hyper-redundant, or snake-like robots. Inextensible curve and surface flows also arise in the context of many problems in computer vision [11], [18] and computer animation [7], and even structural mechanics [21].

Firstly, Kwon and Park studied inextensible flows of curves and developable surfaces, which its arclength is preserved, in Euclidean 3-space [16]. Inextensible flows of curves are studied in many different spaces. Gürbüz have examined inextensible flows of spacelike, timelike and null curves in [9]. After this work Ögrenmiş et al. have studied inextensible curves in Galilean space [20] and Yıldız et al. have studied inextensible flows of curves according to Darboux frame in Euclidean 3-space [22]. Moreover Latifi et al. (2008) studied inextensible flows of curves in Minkowski 3-space [17].

In [4], [5], [12] and [13] the authors focused on timelike and space like curves in E_1^3 and E_1^4 . In the recent work [23], [24] Ö.G. Yıldız et al. gave necessary and sufficient conditions for inextensible flows of non-null curves in E^n and E_1^n .

More generally, from the differential geometric point of view, the study of null curves has its own geometric interest. Many of the classical results from Riemannian geometry have Lorentz counterparts. In fact, spacelike curves or timelike curves can be studied by a similar approach to that in positive definite Riemannian geometry. However, null curves have very different properties from spacelike or timelike curves. In other words, null curve theory has many results which have no Riemannian analogues. The presence of null curves often causes

important and interesting differences, as will be the case in the present study [2].

Nowadays, many important and intensive studies are seen about null curves in Minkowski space. Papers in [1], [3],[8],[14], [15] show that how important field of interest null curves have and obtained some new characterization of this curve in Minkowski space.

In the present paper following [13], [16],[24], we define inextensible flows of partially null and pseudo curves in E_1^4 . We give necessary and sufficient conditions for inextensible flows of partially null and pseudo null curves in E_1^4 .

2. Preliminaries

Let E_1^4 denote the 4-dimensional Minkowski space-time i.e. the Euclidean space E^4 with the standard flat metric given by

$$\langle , \rangle = -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2,$$

where (x_1, x_2, x_3, x_4) is a rectangular coordinate system of E_1^4 . Since g is an indefinite metric, recall that a vector v in E_1^4 can have one of three casual characters: it can be spacelike if $\langle v, v \rangle > 0$ or $v = 0$, timelike if $\langle v, v \rangle < 0$ and null (lightlike) if $\langle v, v \rangle = 0$ and $v \neq 0$. The norm of a vector v is given by $\|v\| = \sqrt{|\langle v, v \rangle|}$. Therefore, v is unit vector if $\langle v, v \rangle = \mp 1$. Next, vectors v and w are said to be orthogonal if $\langle v, w \rangle = 0$. Similarly an arbitrary curve $\gamma(s)$ can be locally spacelike, timelike or null (lightlike), if all of its velocity $\gamma'(s)$ are respectively spacelike, timelike or null (lightlike). Next $\gamma(s)$ is a unit speed curve if $\langle \gamma'(s), \gamma'(s) \rangle = \mp 1$.

Recall that a spacelike curve in E_1^4 is called pseudo null curve or partially null curve, if its principal normal vector is null or its first binormal is null, respectively [10]. A null curve γ is parametrized by arclength function s , if $\langle \gamma''(s), \gamma''(s) \rangle = 1$. In particular, pseudo null curve or partially null curve $\gamma(s)$ has unit speed, if $\langle \gamma'(s), \gamma'(s) \rangle = 1$.

In the following we use the notations and concepts from [10], unless otherwise stated.

Let $\{T, N, B_1, B_2\}$ be the moving Frenet frame along a curve γ in E_1^4 , consisting of the tangent, the principal normal, the first binormal and the second binormal vector fields. Depending on the causal character of γ , the Frenet equations have the following forms.

Case (a). If γ is partially null curve, the Frenet formulas gave as ([10]):

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ -k_1 & 0 & k_2 & 0 \\ 0 & 0 & k_3 & 0 \\ 0 & -k_2 & 0 & -k_3 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix} \quad (2.1)$$

where the third curvature $k_3(s) = 0$ for each s . Such curve has two curvatures $k_1(s)$ and $k_2(s)$ and lies fully in a lightlike hyperplane of E_1^4 . In particular, the following equations hold

$$\langle T, T \rangle = \langle N, N \rangle = 1, \langle B_1, B_1 \rangle = \langle B_2, B_2 \rangle = 0, \quad (2.2)$$

$$\langle T, N \rangle = \langle T, B_1 \rangle = \langle T, B_2 \rangle = \langle N, B_1 \rangle = \langle N, B_2 \rangle = 0, \langle B_1, B_2 \rangle = 1. \quad (2.3)$$

Case (b). If γ is pseudo null curve, the Frenet formulas are ([10]):

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ 0 & 0 & k_2 & 0 \\ 0 & k_3 & 0 & -k_2 \\ -k_1 & 0 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix}, \quad (2.4)$$

where the first curvature $k_1(s) = 0$, if γ is straight line, or $k_1(s) = 1$ in all other cases. Such curve has two curvatures $k_2(s)$ and $k_3(s)$ and the following conditions are satisfied:

$$\langle T, T \rangle = \langle B_1, B_1 \rangle = 1, \langle N, N \rangle = \langle B_2, B_2 \rangle = 0, \quad (2.5)$$

$$\langle T, N \rangle = \langle T, B_1 \rangle = \langle T, B_2 \rangle = \langle N, B_1 \rangle = \langle B_1, B_2 \rangle = 0, \langle N, B_2 \rangle = 1. \quad (2.6)$$

3. Inextensible Flows of partially null curve in E_1^4

Unless otherwise stated we assume that

$$\gamma : [0, l] \times [0, w) \rightarrow E_1^4$$

is a one parameter family of smooth partially null or pseudo null curves in E_1^4 , where l is the arclength of the initial curve. Suppose that u is the curve parametrization variable, $0 \leq u \leq l$. If the speed partially null or pseudo null curves γ is given by $v = \left\| \frac{\partial \gamma}{\partial u} \right\|$, then the arclength of γ is given as a function of u by

$$s(u) = \int_0^u \left\| \frac{\partial \gamma}{\partial u} \right\| du = \int_0^u v du, \quad (3.1)$$

where

$$\left\| \frac{\partial \gamma}{\partial u} \right\| = \sqrt{\left| \left\langle \frac{\partial \gamma}{\partial u}, \frac{\partial \gamma}{\partial u} \right\rangle \right|}.$$

The operator $\frac{\partial}{\partial s}$ is given by

$$\frac{\partial}{\partial s} = \frac{1}{v} \frac{\partial}{\partial u}, \quad (3.2)$$

where $v = \left\| \frac{\partial \gamma}{\partial u} \right\|$.

In this case; the arclength is as follows $ds = v du$.

Definition 3.1. Let γ be a partially null or pseudo null curves in E_1^4 and $\{T, N, B_1, B_2\}$ be the Frenet frame of γ in Minkowski space-time. Any flow of the partially null or pseudo null curves can be expressed as follows

$$\frac{\partial \gamma}{\partial t} = \beta_1 T + \beta_2 N + \beta_3 B_1 + \beta_4 B_2. \quad (3.3)$$

where, β_i is the i^{th} scalar speed of the partially null curves γ .

Let the arclength variation be

$$s(u, t) = \int_0^u v \, du.$$

In E_1^4 , the requirement that the partially null or pseudo null curves not be subject to any elongation or compression can be expressed by the condition

$$\frac{\partial}{\partial t} s(u, t) = \int_0^u \frac{\partial v}{\partial t} \, du = 0,$$

where $u \in [0, l]$.

Definition 3.2. Let γ be a partially null or pseudo null curves in E_1^4 . A partially null curve evolution $\gamma(u, t)$ and its flow $\frac{\partial \gamma}{\partial t}$ are said to be inextensible if

$$\frac{\partial}{\partial t} \left\| \frac{\partial \gamma}{\partial u} \right\| = 0. \quad (3.4)$$

Before deriving the necessary and sufficient condition for inelastic partially null or pseudo null curves flow, we need the following lemma.

Lemma 3.3. Let $\{T, N, B_1, B_2\}$ be the Frenet frame of a partially null curve γ and

$$\frac{\partial \gamma}{\partial t} = \beta_1 T + \beta_2 N + \beta_3 B_1 + \beta_4 B_2$$

be a smooth flow of a partially null curves γ in E_1^4 . If $\gamma(s)$ be unit speed partially null curve in E_1^4 , with the curvature $k_3(s) = 0$, then we have the following equality

$$\frac{\partial v}{\partial t} = \left(\frac{\partial \beta_1}{\partial u} - \beta_2 k_1 v \right). \quad (3.5)$$

Proof. Suppose that $\frac{\partial \gamma}{\partial t}$ be a smooth flow of the partially null curve in E_1^4 , with the curvature $k_3(s) = 0$. Using definition of γ , we have

$$v^2 = \left\langle \frac{\partial \gamma}{\partial u}, \frac{\partial \gamma}{\partial u} \right\rangle. \quad (3.6)$$

Then, by differentiating (3.6), we get

$$2v \frac{\partial v}{\partial t} = \frac{\partial}{\partial t} \left\langle \frac{\partial \gamma}{\partial u}, \frac{\partial \gamma}{\partial u} \right\rangle.$$

On the other hand, as $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial t}$ commute, we have

$$v \frac{\partial v}{\partial t} = \left\langle \frac{\partial \gamma}{\partial u}, \frac{\partial}{\partial u} \left(\frac{\partial \gamma}{\partial t} \right) \right\rangle.$$

From (3.3), we obtain

$$v \frac{\partial v}{\partial t} = \left\langle \frac{\partial \gamma}{\partial u}, \frac{\partial}{\partial u} (\beta_1 T + \beta_2 N + \beta_3 B_1 + \beta_4 B_2) \right\rangle .$$

By using (2.1), we have

$$\begin{aligned} \frac{\partial v}{\partial t} = & \left\langle T, \left(\frac{\partial \beta_1}{\partial u} - k_1 \beta_2 v \right) T + \left(\frac{\partial \beta_2}{\partial u} + \beta_1 k_1 v - \beta_4 k_2 v \right) N + \right. \\ & \left. \left(\frac{\partial \beta_3}{\partial u} + k_2 \beta_2 v \right) B_1 + \frac{\partial \beta_4}{\partial u} B_2 \right\rangle . \end{aligned} \quad (3.7)$$

This clearly forces

$$\frac{\partial v}{\partial t} = \left(\frac{\partial \beta_1}{\partial u} - \beta_2 k_1 v \right) .$$

Lemma 3.4. Let $\{T, N, B_1, B_2\}$ be the Frenet frame of a partially null curves γ and $\frac{\partial \gamma}{\partial t} = \beta_1 T + \beta_2 N + \beta_3 B_1 + \beta_4 B_2$ be a smooth flow of a partially null curves γ in E_1^4 . If $\gamma(s)$ is a unit speed partially null curve in E_1^4 , with the curvature $k_3(s) = 0$, then we have the following equality

$$\frac{\partial \beta_1}{\partial u} = \beta_2 k_1 v. \quad (3.8)$$

Proof. Let us assume that the partially null curve flow is inextensible. From (3.4), we have

$$\frac{\partial}{\partial t} s(u, t) = \int_0^u \frac{\partial v}{\partial t} du = \int_0^u \left(\frac{\partial \beta_1}{\partial u} - \beta_2 k_1 v \right) du = 0. \quad (3.9)$$

This clearly forces

$$\frac{\partial \beta_1}{\partial u} - \beta_2 k_1 v = 0.$$

We now restrict ourselves to arc length parametrized curves. That is, $v = 1$ and the local coordinate u corresponds to the curve arc length s . Then, we have the following lemma.

Lemma 3.5. Let $\{T, N, B_1, B_2\}$ be the Frenet frame of a partially null curves γ and $\frac{\partial \gamma}{\partial t} = \beta_1 T + \beta_2 N + \beta_3 B_1 + \beta_4 B_2$ be a smooth flow of a partially null curves γ in E_1^4 . The differentiations of $\{T, N, B_1, B_2\}$ with respect to t is

$$\frac{\partial T}{\partial t} = \left(\frac{\partial \beta_2}{\partial s} + \beta_1 k_1 - \beta_4 k_2 \right) N + \left(\frac{\partial \beta_3}{\partial s} + \beta_2 k_2 \right) B_1 + \frac{\partial \beta_4}{\partial s} B_2, \quad (3.10)$$

$$\frac{\partial N}{\partial t} = - \left(\frac{\partial \beta_2}{\partial s} + \beta_1 k_1 - \beta_4 k_2 \right) T + \psi_2 B_1 + \psi_1 B_2, \quad (3.11)$$

$$\frac{\partial B_1}{\partial t} = - \frac{\partial \beta_4}{\partial s} T - \psi_1 N + \psi_3 B_1, \quad (3.12)$$

$$\frac{\partial B_2}{\partial t} = - \left(\frac{\partial \beta_3}{\partial s} + k_2 \beta_2 \right) T - \psi_2 N + \psi_3 B_2, \quad (3.13)$$

where

$$\psi_1 = \langle \frac{\partial N}{\partial t}, B_1 \rangle, \quad \psi_2 = \langle \frac{\partial N}{\partial t}, B_2 \rangle, \quad \psi_3 = \langle \frac{\partial B_1}{\partial t}, B_2 \rangle.$$

Proof. From the assumption, we have

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial t} \frac{\partial \gamma}{\partial s} = \frac{\partial}{\partial s} (\beta_1 T + \beta_2 N + \beta_3 B_1 + \beta_4 B_2).$$

Thus, it is seen that

$$\begin{aligned} \frac{\partial T}{\partial t} &= \left(\frac{\partial \beta_1}{\partial s} + k_1 \beta_2 \right) T + \left(\frac{\partial \beta_2}{\partial s} + \beta_1 k_1 - \beta_4 k_2 \right) N \\ &\quad + \left(\frac{\partial \beta_3}{\partial s} + k_2 \beta_2 \right) B_1 + \frac{\partial \beta_4}{\partial s} B_2. \end{aligned} \quad (3.14)$$

Substituting (3.8) into (3.14), we get (3.10).

Since

$$\begin{aligned} \langle T, N \rangle &= 0 \Rightarrow \langle T, \frac{\partial N}{\partial t} \rangle = - \left(\frac{\partial \beta_2}{\partial s} + \beta_1 k_1 - \beta_4 k_2 \right), \\ \langle T, B_1 \rangle &= 0 \Rightarrow \langle T, \frac{\partial B_1}{\partial t} \rangle = - \frac{\partial \beta_4}{\partial s}, \\ \langle T, B_2 \rangle &= 0 \Rightarrow \langle T, \frac{\partial B_2}{\partial t} \rangle = - \left(\frac{\partial \beta_1}{\partial s} + k_1 \beta_2 \right), \\ \langle N, B_1 \rangle &= 0 \Rightarrow \langle N, \frac{\partial B_1}{\partial t} \rangle = -\psi_1, \\ \langle N, B_2 \rangle &= 0 \Rightarrow \langle N, \frac{\partial B_2}{\partial t} \rangle = -\psi_2, \\ \langle B_1, B_2 \rangle &= 1 \Rightarrow \langle B_1, \frac{\partial B_2}{\partial t} \rangle = -\psi_3, \end{aligned}$$

we have

$$\langle N, \frac{\partial N}{\partial t} \rangle = \langle B_1, \frac{\partial B_1}{\partial t} \rangle = \langle B_2, \frac{\partial B_2}{\partial t} \rangle = 0.$$

In a similar manner above, we can obtain (3.11), (3.12) and (3.13).

Theorem 3.6. Let $\{T, N, B_1, B_2\}$ be the Frenet frame of a partially null curves γ and $\frac{\partial \gamma}{\partial t} = \beta_1 T + \beta_2 N + \beta_3 B_1 + \beta_4 B_2$ be a smooth flow of a partially null curves γ in E_1^4 . Then, there exists the following system of partial differential equation.

$$\frac{\partial k_1}{\partial t} = \frac{\partial^2 \beta_2}{\partial s^2} + \frac{\partial (\beta_1 k_1)}{\partial s} - \frac{\partial (\beta_4 k_2)}{\partial s} - \frac{\partial \beta_4}{\partial s} k_2$$

Proof From Lemma 3.4, we have

$$\begin{aligned} \frac{\partial}{\partial s} \frac{\partial T}{\partial t} &= \left(\frac{\partial^2 \beta_2}{\partial s^2} + \frac{\partial(\beta_1 k_1)}{\partial s} - \frac{\partial(\beta_4 k_2)}{\partial s} \right) N + \left(\frac{\partial \beta_2}{\partial s} + \beta_1 k_1 - \beta_4 k_2 \right) (-k_1 T + k_2 B_1) \\ &\quad \left(\frac{\partial^2 \beta_3}{\partial s^2} + \frac{\partial(\beta_2 k_2)}{\partial s} \right) B_1 + \frac{\partial^2 \beta_4}{\partial s^2} B_2 + \frac{\partial \beta_4}{\partial s} (-k_2 N). \end{aligned}$$

Then

$$\begin{aligned} \frac{\partial}{\partial s} \frac{\partial T}{\partial t} &= - \left(\frac{\partial \beta_2}{\partial s} k_1 + \beta_1 k_1^2 - \beta_4 k_1 k_2 \right) T + \left(\frac{\partial^2 \beta_2}{\partial s^2} + \frac{\partial(\beta_1 k_1)}{\partial s} - \frac{\partial(\beta_4 k_2)}{\partial s} - \frac{\partial \beta_4}{\partial s} k_2 \right) N \\ &\quad + \left(\frac{\partial^2 \beta_3}{\partial s^2} + \frac{\partial(\beta_2 k_2)}{\partial s} + \frac{\partial \beta_2}{\partial s} k_2 + \beta_1 k_1 k_2 - \beta_4 k_2^2 \right) B_1 + \frac{\partial^2 \beta_4}{\partial s^2} B_2. \end{aligned} \quad (3.15)$$

Note that

$$\frac{\partial}{\partial t} \left(\frac{\partial T}{\partial s} \right) = \frac{\partial k_1}{\partial t} N - \left(\frac{\partial \beta_2}{\partial s} k_1 + \beta_1 k_1^2 - \beta_4 k_1 k_2 \right) T + \psi_2 k_1 B_1 + \psi_1 k_1 B_2. \quad (3.16)$$

Hence from (3.15) and (3.16), we get

$$\frac{\partial k_1}{\partial t} = \frac{\partial^2 \beta_2}{\partial s^2} + \frac{\partial(\beta_1 k_1)}{\partial s} - \frac{\partial(\beta_4 k_2)}{\partial s} - \frac{\partial \beta_4}{\partial s} k_2.$$

This completes the proof.

Corollary 3.7. Let $\{T, N, B_1, B_2\}$ be the Frenet frame of a partially null curves γ and $\frac{\partial \gamma}{\partial t} = \beta_1 T + \beta_2 N + \beta_3 B_1 + \beta_4 B_2$ be a smooth flow of a partially null curves γ in E_1^4 . Then, we have the following equalities.

$$k_1 = \frac{1}{\psi_1} \left[\frac{\partial^2 \beta_4}{\partial s^2} \right],$$

$$\frac{\partial^2 \beta_3}{\partial s^2} + \frac{\partial(\beta_2 k_2)}{\partial s} + \frac{\partial \beta_2}{\partial s} k_2 + \beta_1 k_1 k_2 - \beta_4 k_2^2 - \psi_2 k_1 = 0.$$

Theorem 3.8. Let $\{T, N, B_1, B_2\}$ be the Frenet frame of a partially null curves γ and $\frac{\partial \gamma}{\partial t} = \beta_1 T + \beta_2 N + \beta_3 B_1 + \beta_4 B_2$ be a smooth flow of a partially null curves γ in E_1^4 . Then, we have

$$k_1 = - \left[\frac{\partial \psi_1}{\partial s} \not\! / \frac{\partial \beta_4}{\partial s} \right] \quad (3.17)$$

and

$$k_2 = \frac{1}{\psi_1} \left[\frac{\partial \psi_3}{\partial s} \right]. \quad (3.18)$$

Proof. Noting that $\frac{\partial}{\partial s} \left(\frac{\partial B_1}{\partial t} \right) = \frac{\partial}{\partial t} \left(\frac{B_1}{\partial s} \right)$, we have the equations (3.17) and (3.18).

Theorem 3.9. Let $\{T, N, B_1, B_2\}$ be the Frenet frame of a partially null curves γ and $\frac{\partial \gamma}{\partial t} = \beta_1 T + \beta_2 N + \beta_3 B_1 + \beta_4 B_2$ be a smooth flow of a partially null curves γ in E_1^4 . Then, there exists the following system of partially differential equation.

$$\frac{\partial k_2}{\partial t} = \frac{\partial \psi_2}{\partial s} + \frac{\partial \beta_3}{\partial s} k_1 - \beta_2 k_1 k_2 - \psi_3 k_2. \quad (3.19)$$

Proof. By the same way above and considering $\frac{\partial}{\partial s} \left(\frac{\partial B_2}{\partial t} \right) = \frac{\partial}{\partial t} \left(\frac{B_2}{\partial s} \right)$ we reach.

$$\frac{\partial k_2}{\partial t} = \frac{\partial \psi_2}{\partial s} + \frac{\partial \beta_3}{\partial s} k_1 - \beta_2 k_1 k_2 - \psi_3 k_2.$$

4. Inextensible Flows of pseudo null curve in E_1^4

We omit the proofs of the following theorems because of having close analogy of the theorems given above.

Lemma 4.1. Let $\{T, N, B_1, B_2\}$ be the Frenet frame of a pseudo null curve γ and

$$\frac{\partial \gamma}{\partial t} = \alpha_1 T + \alpha_2 N + \alpha_3 B_1 + \alpha_4 B_2$$

be a smooth flow of a pseudo null curve γ in E_1^4 . If $\gamma(s)$ be unit speed pseudo null curve in E_1^4 , with curvature $k_1(s) = 1$, $k_2(s)$ and $k_3(s) \neq 0$, then we have the following equality

$$\frac{\partial v}{\partial t} = \left(\frac{\partial \alpha_1}{\partial u} - \alpha_4 v \right).$$

Lemma 4.2. Let $\{T, N, B_1, B_2\}$ be the Frenet frame of a pseudo null curve γ and $\frac{\partial \gamma}{\partial t} = \alpha_1 T + \alpha_2 N + \alpha_3 B_1 + \alpha_4 B_2$ be a smooth flow of a pseudo null curve γ in E_1^4 . If $\gamma(s)$ be unit speed pseudo null curve in E_1^4 , with curvature $k_1(s) = 1$, $k_2(s)$ and $k_3(s) \neq 0$, then we have the following equality

$$\frac{\partial \alpha_1}{\partial u} = \alpha_4 v.$$

Lemma 4.3 Let $\{T, N, B_1, B_2\}$ be the Frenet frame of a pseudo null curve γ and $\frac{\partial \gamma}{\partial t} = \alpha_1 T + \alpha_2 N + \alpha_3 B_1 + \alpha_4 B_2$ be a smooth flow of a pseudo null curve γ in E_1^4 . The differentiations of $\{T, N, B_1, B_2\}$ with respect to t is

$$\frac{\partial T}{\partial t} = \left(\frac{\partial \alpha_2}{\partial s} + \alpha_1 + \alpha_3 k_3 \right) N + \left(\frac{\partial \alpha_3}{\partial s} + \alpha_2 k_2 - \alpha_4 k_3 \right) B_1 + \left(\frac{\partial \alpha_4}{\partial s} - \alpha_3 k_2 \right) B_2,$$

$$\frac{\partial N}{\partial t} = - \left(\frac{\partial \alpha_4}{\partial s} - \alpha_3 k_2 \right) T + \psi_2 N + \psi_1 B_1,$$

$$\frac{\partial B_1}{\partial t} = - \left(\frac{\partial \alpha_3}{\partial s} + \alpha_2 k_2 - \alpha_4 k_3 \right) T + \psi_3 N - \psi_1 B_2,$$

$$\frac{\partial B_2}{\partial t} = - \left(\frac{\partial \alpha_2}{\partial s} + \alpha_1 + \alpha_3 k_3 \right) T - \psi_3 B_1 - \psi_2 B_2,$$

where

$$\psi_1 = \langle \frac{\partial N}{\partial t}, B_1 \rangle, \quad \psi_2 = \langle \frac{\partial N}{\partial t}, B_2 \rangle, \quad \psi_3 = \langle \frac{\partial B_1}{\partial t}, B_2 \rangle.$$

Theorem 4.4. Let $\{T, N, B_1, B_2\}$ be the Frenet frame of a pseudo null curve γ and $\frac{\partial \gamma}{\partial t} = \alpha_1 T + \alpha_2 N + \alpha_3 B_1 + \alpha_4 B_2$ be a smooth flow of a pseudo null curve γ in E_1^4 . Then, there exists the following system of partially differential equations

$$\frac{\partial^2 \alpha_3}{\partial s^2} + \frac{\partial (\alpha_2 k_2)}{\partial s} - \frac{\partial (\alpha_4 k_3)}{\partial s} - \frac{\partial \alpha_4}{\partial s} k_3 + \frac{\partial \alpha_2}{\partial s} k_2 + \alpha_1 k_2 + 2\alpha_3 k_3 k_2 - \psi_1 = 0,$$

$$\frac{\partial^2 \alpha_2}{\partial s^2} + \frac{\partial \alpha_1}{\partial s} + \frac{\partial (\alpha_3 k_3)}{\partial s} + \frac{\partial \alpha_3}{\partial s} k_3 + \alpha_2 k_2 k_3 - \alpha_4 k_3^2 - \psi_2 = 0,$$

$$\frac{\partial^2 \alpha_4}{\partial s^2} - \frac{\partial (\alpha_3 k_2)}{\partial s} - \frac{\partial \alpha_3}{\partial s} k_2 - \alpha_2 k_2 + \alpha_4 k_2 k_3 = 0,$$

$$\frac{\partial k_3}{\partial s} = \frac{\partial \psi_3}{\partial s} - \frac{\partial \alpha_3}{\partial s} - \alpha_2 k_2 + \alpha_4 k_3 - \psi_2 k_3,$$

$$\frac{\partial k_2}{\partial s} = \frac{\partial \psi_1}{\partial s} + \psi_2 k_2,$$

$$\frac{\partial \psi_2}{\partial s} = \frac{\partial \alpha_4}{\partial s} - \alpha_3 k_2 - \psi_1 k_3 + \psi_2 k_3.$$

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